# Charge-induced instability and macroscopic quantum nucleation phenomena at the crystal <sup>4</sup>He facet

S. N. Burmistrov Kurchatov Institute, 123182 Moscow, Russia

An existence of the charge-induced instability is well known for the <sup>4</sup>He crystal surface in the rough state. Much less is known about charge-induced instability at the <sup>4</sup>He crystal surface in the smooth well-faceted state below the roughening transition temperature. To meet the lack, we examine here the latter case. As long as the electric field normal to the crystal facet is below the critical value same as for the rough surface, the crystal faceting remains absolutely stable. Above the critical field, unlike absolutely unstable state of the rough surface, the crystal facet crosses over to the metastable state separated from new crushed state with a potential barrier proportional to the square of the linear facet step energy. The onset and development of the instability at the charged crystal facet has much in common with the nucleation kinetics of first-order phase transitions. Depending on the temperature, the electric breaking strength is determined either by thermal activation at high temperatures or by quantum tunneling at sufficiently low temperatures.

### PACS numbers: 67.80.-s, 64.60.Q-

#### I. INTRODUCTION

It is well known that a charged interface between two fluids can develop an electrohydrodynamic instability at sufficiently high density of charges. Such charge-induced instability results from the competition between the electric forces of like charges and forces of surface tension and gravity. Naturally, the liquid phases of helium have become one of physical systems for the theoretical and experimental studies of electrohydrodynamic instabilities [1], in particular, as softening of the gravitational-capillary wave spectrum [2–4], charge-induced deformations of the interface [5–7], formation of regular array of dimples [5–7], individual multielectron dimples [8], and hexagonal reconstructuring [9].

It is of particular interest to compare the onset and development of the electrohydrodynamic instability at the liquid-solid <sup>4</sup>He interface with that at the interface between two fluids. The first theoretical and experimental studies have shown that a charged-induced instability at the superfluid-solid <sup>4</sup>He looks roughly like the electrohydrodynamic instability at the free liquid <sup>4</sup>He surface [10–13]. For the high temperature region where the crystal surface is in the rough state, such similar behavior is expectable since the superfluid-solid interface in the rough state has a very high mobility and interface excitations represent weakly damping crystallization waves whose dispersion [14, 15] is quite similar to that of usual gravitational-capillary waves at the free liquid surface.

To date, no systematic study has been made on the onset and development of electrohydrodynamic instability at the well-faceted and atomically smooth crystal surfaces which may have an infinitely large stiffness and excitation spectrum differing from the usual crystallization wave spectrum. The most striking distinction of the smooth faceted crystal surface from the rough one is the existence of nonanalytic cusplike behavior in the angle dependence for the surface tension, e.g., [16–18]. The ori-

gin of the singularity is directly connected with nonzero magnitude of the facet step energy below the roughening transition temperature of about 1.2 K.

In present work we attempt the electrohydrodynamic instability at the smooth faceted surface of a <sup>4</sup>He crystal in contact with its liquid phase. As we will see below, the close similarity between the rough and smooth states of the crystal surfaces extends until the charge density is below the critical one and the state and shape of the surface are stable. As the charge density increases, the development of the instabilities becomes different in kind. Unlike the rough crystal surface, the faceted surface crosses over to a metastable state and the further development of the instability is accompanied by overcoming some nucleation barrier. The barrier height is proportional to the square of the facet step energy and drastically reduces as the charge density increases. At the sufficiently low temperatures the thermal activation mechanism of overcoming the barrier is replaced with the quantum tunneling through the nucleation barrier. On the whole, the charge-induced reconstructuring of the faceted crystal surface resembles much first-order phase transitions and macroscopic quantum nucleation phenomena in the helium systems [19–23].

For simplicity, we keep in mind the basal plane of hexagonal <sup>4</sup>He crystal as an example of the crystal facet and neglect any anisotropy in the plane. We also suppose that the temperature is below the roughening transition temperature and the crystal surface is well-defined and faceted.

# II. HAMILTONIAN. THE ONSET OF INSTABILITY AT THE CRYSTAL FACET

Let us assume that the crystal surface is parallel to the xy plane, with vertical position at z=0. In order to consider the stability of the surface, we proceed as follows. First, we call  $\zeta = \zeta(\mathbf{r})$  the displacement of the surface

from its horizontal position z=0 with  ${\boldsymbol r}=(x,y)$  as a two-dimensional radius-vector. In addition to the surface tension force and the force of gravity due to difference in the densities between the solid and liquid states  $\Delta \rho$ , one should involve also the interaction of the charges with electric field E and the direct Coulomb interaction between the charges. Then the total energy U of a charged surface can be written as

$$U = \int d^2r \left( \alpha(\boldsymbol{\nu}) \sqrt{1 + (\nabla \zeta)^2} + \Delta \rho g \frac{\zeta^2}{2} + eEn(\boldsymbol{r}) \zeta \right) + \frac{1}{2} \iint d^2r \, d^2r' \, \frac{en(\boldsymbol{r})en(\boldsymbol{r}')}{|\boldsymbol{l} - \boldsymbol{l}'|}. \tag{1}$$

Here  $\boldsymbol{l}=(\boldsymbol{r},\zeta)$  stands for the three-dimensional coordinate of a point at the surface,  $n(\boldsymbol{r})$  is the density of electrons with charge e, g is the acceleration of gravity, and  $\alpha(\boldsymbol{\nu})$  is the energy of a unit surface area or surface tension.

Unlike the fluid-fluid interface, the surface tension coefficient  $\alpha(\nu)$  for the crystal facet depends essentially on the direction of the normal  $\nu$  to the interface against crystallographic axes. In our simplest case this is a function of angle  $\vartheta$  alone between the normal and the crystallographic [0001] or c-axis of the crystal hcp structure with the geometric relation  $|\tan\vartheta|=|\nabla\zeta|$ .

For the crystal facet tilted by small angle  $\vartheta$  from the basal plane, the expansion of surface tension  $\alpha(\vartheta)$  usually written, e.g. [16–18], as

$$\alpha(\vartheta) = (\alpha_0 + \alpha_1 \tan |\vartheta| + \ldots) \cos \vartheta, \quad |\tan \vartheta| = |\nabla \zeta|,$$

can be represented for the small angles by a series

$$\alpha(\vartheta) = \alpha(0) + \alpha_1 \mid \vartheta \mid +\alpha''(0) \frac{\vartheta^2}{2} + \dots, \quad \mid \vartheta \mid \ll 1.$$

We intentionally do not write the next terms of expansion, e.g., cubic one due to step-step interaction, since we assume to study only small bending of the crystal surface. The angular behavior has a nonanalytic cusp-like behavior at  $\vartheta=0$  due to  $\alpha_1=\alpha_1(T)$  representing a ratio of the linear facet step energy  $\beta$  to the crystallographic interplane spacing. Below the roughening transition temperature for the basal plane  $T_R \sim 1.2$  K the facet step energy  $\beta=\beta(T)$  is positive and vanishes for temperatures  $T>T_R$ .

To determine the equilibrium shape of the surface  $\zeta(r)$  and equilibrium charge distribution n(r), we must minimize the energy functional (1) against  $\zeta$  and n at a given total surface charge Q. This condition can readily be taken into account by augmenting the energy functional with the Lagrange factor  $\lambda$  in the form

$$-\lambda \int en(\boldsymbol{r}) d^2r.$$

In addition, treating the energy functional, we naturally imply one more obvious condition  $n(r) \ge 0$ .

In the general form the minimization of the energy functional is a practically unsolvable problem because its expression (1) contains not only quadratic terms in  $\zeta$  and n. Thus, we start first from analyzing small surface bending  $\zeta(r)$  and small gradients  $|\nabla \zeta| \ll 1$ . The latter implies implicitly that  $|\zeta| \ll r$  and we can put approximately l = r in the denominator of the Coulomb term in Eq. (1). Next, we expand the surface tension term in  $|\nabla \zeta|$ , retain the linear and quadratic terms alone, and arrive at the following expression for the total excess energy U associated with nonzero surface bending  $\zeta(r)$ 

$$U = \int d^2r \left( \alpha_1 |\nabla \zeta| + \frac{\alpha_0}{2} (\nabla \zeta)^2 + \frac{\Delta \rho g}{2} \zeta^2 + eEn(\mathbf{r})\zeta \right)$$
$$+ \frac{1}{2} \iint d^2r \, d^2r' \, \frac{en(\mathbf{r})en(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}.$$

Here we have labeled

$$\alpha_0 = \alpha(0) + \alpha''(0)$$

as a surface stiffness.

The spatial scale of surface distortion is usually determined by the capillary length  $\lambda_0 = (\alpha_0/\Delta\rho\,g)^{1/2} \approx 1$  mm if one takes  $\alpha_0 \approx 0.2$  erg/cm² and  $\Delta\rho \approx 0.018$  g/cm³ for  $^4{\rm He}$  [16]. The electric field E and charge surface density en have the same dimensionality and their typical scale is  $(\alpha_0\Delta\rho\,g)^{1/4} \approx 400$  V/cm. Accordingly, the typical electron density equals  $(\alpha_0\Delta\rho\,g)^{1/4}/e \approx 2.8 \times 10^9$  cm $^{-2}$ . The number of electrons  $\pi\alpha_0^{5/4}(\Delta\rho\,g)^{-3/4}$  within the circle of radius  $\lambda_0$  runs to  $10^8$ . And lastly, unit of energy corresponds to  $\alpha_0^2/(\Delta\rho\,g) \approx 2.2 \times 10^{-3}$  erg.

So, if we measure  $\zeta$  and r in units of capillary length  $\lambda_0$ , electric field and charge density in units of  $(\alpha_0 \Delta \rho g)^{1/4}$ , and energy in units of  $\alpha_0^2/(\Delta \rho g)$ , the total excess energy U can be expressed in terms of dimensionless units as

$$U = \int d^2r \left( \frac{\alpha_1}{\alpha_0} |\nabla \zeta| + \frac{(\nabla \zeta)^2}{2} + \frac{\zeta^2}{2} + En(\mathbf{r})\zeta \right) + \frac{1}{2} \iint d^2r \, d^2r' \, \frac{n(\mathbf{r})n(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{2}$$

As for the step energy  $\alpha_1$ , we assume that its low temperature value [16] is approximately  $\alpha_1 \approx 0.014 \, \mathrm{erg/cm^2}$ . This value amounts to one-tenth of surface stiffness  $\alpha_0$  and in the following we can keep inequality  $\alpha_1/\alpha_0 \ll 1$  in mind. Moreover, this small parameter justifies all approximations that will be made further.

The uniform state of the surface holds for the electric field values as long as the contribution to the excess energy (2) due to variations  $\zeta(\mathbf{r})$  from  $\zeta=0$  and  $n(\mathbf{r})$  from homogeneous value  $\bar{n}$  is a positive-definite quantity. Using the following equation for Lagrange factor

$$E\zeta(\boldsymbol{r}) + \int \frac{n(\boldsymbol{r}') \, d^2r'}{|\boldsymbol{r} - \boldsymbol{r}'|} = \lambda$$

and putting  $\delta n(\mathbf{r}) = n(\mathbf{r}) - \bar{n}$ , we find for the variation

of the excess energy

$$\delta U = \int d^2r \left( \frac{\alpha_1}{\alpha_0} |\nabla \zeta| + \frac{(\nabla \zeta)^2}{2} + \frac{\zeta^2}{2} + E \, \delta n(\mathbf{r}) \zeta \right) + \frac{1}{2} \iint d^2r \, d^2r' \, \frac{\delta n(\mathbf{r}) \delta n(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{3}$$

To analyze it, we use the Fourier representation

$$\zeta(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \zeta_{\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{r}} \ \text{ and } \ \delta n(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \delta n_{\boldsymbol{k}} e^{i\boldsymbol{k}\boldsymbol{r}},$$

and rewrite the energy variation as

$$\begin{split} \delta U &= \int \! d^2 r \, \frac{\alpha_1}{\alpha_0} |\nabla \zeta| + \\ &\frac{1}{2} \sum_{\pmb{k}} \left[ (k^2 + 1) \zeta_{\pmb{k}} \zeta_{\pmb{k}}^* + E(\delta n_{\pmb{k}} \zeta_{\pmb{k}}^* + \delta n_{\pmb{k}}^* \zeta_{\pmb{k}}) + \frac{2\pi}{k} \delta n_{\pmb{k}} \delta n_{\pmb{k}}^* \right]. \end{split}$$

Minimizing  $\delta U$  over  $\delta n_{k}$  yields the optimum relation

$$\delta n_{\mathbf{k}} = -\frac{k}{2\pi} \zeta_{\mathbf{k}} \tag{4}$$

and the corresponding optimum value of the energy

$$\delta U = \int \frac{\alpha_1}{\alpha_0} |\nabla \zeta| \, d^2 r + \frac{1}{2} \sum_{\mathbf{k}} \left( k^2 + 1 - \frac{kE^2}{2\pi} \right) |\zeta_{\mathbf{k}}|^2. \quad (5)$$

The second term is always positive provided the inequality  $E^2 < 2\pi(k+1/k)$  is satisfied for all wave vectors k. The minimum of the right-hand side of the inequality occurs at  $k=k_c=1$  and corresponds to the critical field  $E_c=\sqrt{4\pi}$ . Thus, the crystal facet is absolutely stable at  $E < E_c$ .

At  $E > E_c$  the stability is lost and the distortions of the homogeneous state should appear. In this regard the situation resembles the loss of stability for the rough state of the crystal surface. However, the development of the stability and the transition to unhomogeneous state differ drastically. In fact, due to positive  $\alpha_1 > 0$  term linear in  $|\nabla \zeta|$  the evolution of the crystal facet perturbations should inevitably be accompanied with overcoming some potential barrier, the barrier height being dependent on the field strength E. The more the field strength, the less the potential barrier height.

To proceed, let us return to the coordinate representation of Eq. (5)

$$\delta U = \int d^2 r \left( \frac{\alpha_1}{\alpha_0} |\nabla \zeta| + \frac{(\nabla \zeta)^2}{2} + \frac{\zeta^2}{2} \right)$$
$$-\frac{1}{2} \iint d^2 r \, d^2 r' \, \frac{E^2}{(2\pi)^2} \frac{\left( \nabla_{\boldsymbol{r}} \zeta(\boldsymbol{r}) \nabla_{\boldsymbol{r}'} \zeta(\boldsymbol{r}') \right)}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{6}$$

and give a qualitative description of the matter. For this purpose, we employ a variational principle and dimensional analysis of the functional (6). Let us represent the surface distortion  $\zeta(r)$  with the aid of the trial function f(x) in the axially symmetrical form as

$$\zeta(\mathbf{r}) = \zeta f(r/R),\tag{7}$$

where  $\zeta$  is a typical magnitude of distortion and R is its typical size. Then we have

$$\delta U(\zeta, R) = \frac{\alpha_1}{\alpha_0} a|\zeta| R + b \frac{\zeta^2}{2} + c \frac{\zeta^2 R^2}{2} - d \frac{E^2}{4\pi^2} \frac{\zeta^2 R}{2},$$

and the dimensionless factors are given by

$$a = \int_0^\infty |f'(r)| 2\pi r \, dr, \quad b = \int_0^\infty f'^2(r) 2\pi r \, dr,$$

$$c = \int_0^\infty f^2(r) 2\pi r \, dr, \quad d = \int d^2 r \, d^2 r' \frac{\mathbf{r} \cdot \mathbf{r}'}{rr'} \frac{f'(r) f'(r')}{|\mathbf{r} - \mathbf{r}'|}$$

$$= \int_0^\infty dk \left( \int_0^\infty dr \, 2\pi r f'(r) J_1(kr) \right)^2,$$

where  $J_1(x)$  is the Bessel function of the first kind.

As the electric field strength exceeds the value  $E_0 = (8\pi^2\sqrt{bc}/d)^{1/2}$ , there appears a region of  $\zeta$  and R with the negative values of  $\delta U$  separated always from  $\delta U = 0$  at  $\zeta = 0$  with the intermediate positive  $\delta U$  values. Rewriting the excess energy  $\delta U(\zeta, R)$  as

$$\delta U = \frac{1}{2} \left( \frac{8\pi^2 \alpha_1}{\alpha_0} \right)^2 \frac{a^2 b/d^2}{E^4 - E_0^4} + \frac{b}{2} \zeta^2 \left( 1 - \sqrt{\frac{c}{b}} \frac{E^2}{E_0^2} R \right)^2 - \frac{c}{2} \left( \frac{E^2}{E_0^2} - 1 \right) \left( |\zeta| R - \frac{\alpha_1}{\alpha_0} \frac{a/c}{E^4/E_0^4 - 1} \right)^2, \ E_0^2 = \frac{8\pi^2 \sqrt{bc}}{d},$$

one can readily see that the state of the crystal facet changes from the stable to metastable state at  $E>E_0$  and the point

$$|\zeta_0| = \frac{\alpha_1}{\alpha_0} \frac{a}{\sqrt{bc}} \frac{E^2 E_0^2}{E^4 - E_0^4}$$
 and  $R_0 = \sqrt{\frac{b}{c}} \frac{E_0^2}{E^2}$ 

becomes a saddle point of the potential relief. The potential barrier height equal to

$$U_0 = \frac{a^2}{2c} \frac{\alpha_1^2}{\alpha_0^2} \frac{E_0^4}{E^4 - E_0^4}$$

must be overcome to break the flat faceting of a crystal surface.

Unfortunately, we cannot find the exact function f(r) and, correspondingly, values of factors a, b, c and d which optimize the functional (6). However, it is clear that the potential barrier height should be infinitely large at  $E = E_c$  and thus  $E_0 = E_c$  for the exact solution. This entails the obvious relation  $d = 2\pi (bc)^{1/2}$  between coefficients for the exact solution. To estimate them, we use a trial function  $f(x) = \exp(-x^2)$ . The direct calculation results in

$$a=\pi^{3/2}, \ b=\pi, \ c=\pi/2, \ d=\pi^{5/2}/\sqrt{2},$$

and

$$\frac{E_c}{E_0} = \sqrt{\frac{d}{2\pi\sqrt{bc}}} = \frac{\pi^{1/4}}{\sqrt{2}} \approx 0.94$$
 (8)

in place of unity. Hence we may expect an accuracy of our estimate within about 10%.

Let us compare the height  $U_0$  of the potential barrier at the saddle point with the roughening transition temperature  $T_R$  about 1.2 K. In the dimensional units we have

$$U_0 = \frac{a}{2c} \frac{\alpha_1^2}{\Delta \rho g} \frac{E_c^4}{E^4 - E_c^4} \sim 1.4 \times 10^{11} \frac{E_c^4}{E^4 - E_c^4} \text{ (in K)}.$$

One may be surprised with the huge barrier height so that, unlike the rough crystal surface absolutely unstable at  $E \ge E_c$ , tens of  $E_c$  should keep a crystal facet practically stable for an experimentally available time. Provided we expect a reasonable observation time of destructing the faceted state due to thermal activation mechanism, we should provide a ratio  $U_0/T$  of about a few tens [21, 22] This means that the electric field E should exceed the critical one  $E_c$  by a factor of about 300. The same factor certainly refers to the surface density of charges.

In the dimensional CGSE units the bending deflection  $\zeta_0$  and the typical size of inhomogeneity  $R_0$  are given by

$$|\zeta_0| = 4\pi \frac{a}{\sqrt{bc}} \alpha_1 \frac{E^2}{E^4 - E_c^4} \sim 31 \frac{\alpha_1 E^2}{E^4 - E_c^4},$$

$$R_0 = \sqrt{\frac{b}{c}} \frac{4\pi \alpha_0}{E^2} \sim 18 \frac{\alpha_0}{E^2}.$$

In the weak fields of few critical values the critical parameters  $R_0$  and  $|\zeta_0|$  prove to be of macroscopic sizes in accordance with macroscopically large height of the potential barrier.

For  $E=300E_c$ , we find approximately  $R_0\sim 16$  nm and  $|\zeta_0|\sim 2$  nm. On the whole, the electric field should be very large compared with the critical value  $E_c$  in order to reduce significantly the nucleation barrier for the effective production of a few circular crystal terraces tilted with the angle about  $\arctan(\alpha_1/\alpha_0)\sim 4^\circ$ . In this sense the critical fluctuation represents a region of the crystal surface in the rough state.

From the physical point of view the angle of slope  $\arctan(\alpha_1/\alpha_0) \sim 4^\circ$  is determined by the competition of two contributions into the total surface energy. One originates from the regular surface term  $\alpha_0\zeta^2$  and the second does from irregular step tension term  $\alpha_1|\zeta|R$ . Provided  $\alpha_0\zeta^2\gg\alpha_1|\zeta|R$ , the latter contribution becomes negligible and thus the interface properties resemble those in the rough surface state. On the contrary, if  $\alpha_0\zeta^2\ll\alpha_1|\zeta|R$ , the dominant term linear in  $|\zeta|$  is responsible for the origin of a potential barrier since the other terms quadratic in  $\zeta$  are yet insignificant.

Note that the small gradient approximation we use is satisfied since  $|\nabla \zeta| \sim |\zeta_0|/R_0 \sim \alpha_1/\alpha_0 \ll 1$  with the exception of narrow region  $E \sim E_c$ . The latter remark refers also to justifying small density variations  $\delta n \ll \bar{n}$  valid to the extent of smallness  $|\zeta_0|/R_0$ .

### III. LAGRANGIAN. THE QUANTUM BREAKING OF THE CRYSTAL FACET

The destruction of the faceted crystal surface is accompanied by overcoming some potential barrier depending on the charge surface density. There are two basic mechanisms to overcome the potential barrier. One is the thermal activation efficient at high temperatures and the second is the quantum tunneling through a potential barrier dominant at sufficiently low temperatures. In order to treat the quantum tunneling, it is necessary to involve the interface dynamics, in particular, to determine the kinetic energy of the charged interface in addition to the potential energy U.

As a first step, we employ the so-called metallic approximation. In this approximation it is assumed that the mobility of electrons along the superfluid-crystal  $\mathrm{He}^4$  interface is very high and the charged helium interface represents an equipotential surface so that the electric field is always normal to the interface as for a well-conducting metal. A necessary condition for such approximation assumes at least that the plasma oscillation frequency of a two-dimensional layer of electrons with effective mass  $m_e$ 

$$\Omega_p \sim (2\pi n e^2 k/m_e)^{1/2}$$
 (9)

is much larger than the typical frequency  $\omega$  of the gravitational-capillary or melting-crystallization waves at the same wave vector k. So, within our first approximation we believe that the charge density distribution  $n(t, \boldsymbol{r})$  has sufficient time to accommodate to the surface distortion  $\zeta(t, \boldsymbol{r})$  and is determined by the electrostatic relations in accordance with the profile  $\zeta(t, \boldsymbol{r})$ .

Neglecting possible energy dissipation, we describe the charged interface dynamics using the following action

$$S = \int dt L[\zeta(t, \mathbf{r}), \dot{\zeta}(t, \mathbf{r}), n(t, \mathbf{r})]$$
 (10)

with the Lagrangian L equal to the difference between the kinetic energy functional and the potential energy functional U introduced by Eq. (1)

$$L = \frac{\rho_{\text{eff}}}{2} \iint d^2r \, d^2r' \, \frac{\dot{\zeta}(t, \, \boldsymbol{r})\dot{\zeta}(t, \, \boldsymbol{r}')}{2\pi |\boldsymbol{r} - \boldsymbol{r}'|} - U[\zeta(t, \, \boldsymbol{r}), n(t, \, \boldsymbol{r})].$$

Here we ignore the compressibility of the both liquid and solid phases. Because of low temperature consideration we will also neglect the normal component density in the superfluid phase or, equivalently, difference between the superfluid density  $\rho_s$  and the density of the liquid phase  $\rho$ . Then the effective interface density  $\rho_{\rm eff}$  is given by

$$\rho_{\rm eff} = (\rho' - \rho)^2 / \rho \approx 1.9 \, {\rm mg/cm}^3$$

and depends on the difference  $\Delta \rho = \rho' - \rho$  between the solid density  $\rho'$  and the liquid density  $\rho$ . For our purposes, the exact magnitude of the effective density is inessential.

Next, for convenience, let us introduce units of time equal to  $(\rho_{\rm eff}\lambda_0^3/\alpha_0)^{1/2} \approx 3.1$  ms and measure the action in units of  $(\alpha_0\rho_{\rm eff}\lambda_0^7)^{1/2} \approx 0.62 \times 10^{-5}$  erg·s. Using the speculations and arguments bringing us to Eq. (4) and then to Eq. (6), we arrive at examining the following effective action

$$S = \int dt \, L_{\text{eff}}[\zeta(t, \, m{r}), \dot{\zeta}(t, \, m{r})]$$

with the dimensionless Lagrangian

$$L_{\text{eff}} = \frac{1}{2} \iint d^2r \, d^2r' \, \frac{\dot{\zeta}(t, \, \boldsymbol{r}) \dot{\zeta}(t, \, \boldsymbol{r}')}{2\pi |\boldsymbol{r} - \boldsymbol{r}'|}$$

$$- \int d^2r \left( \frac{\alpha_1}{\alpha_0} |\nabla \zeta| + \frac{(\nabla \zeta)^2}{2} + \frac{\zeta^2}{2} \right)$$

$$+ \frac{1}{2} \iint d^2r \, d^2r' \, \frac{E^2}{(2\pi)^2} \frac{\left( \nabla_{\boldsymbol{r}} \zeta(t, \, \boldsymbol{r}) \nabla_{\boldsymbol{r}'} \zeta(t, \, \boldsymbol{r}') \right)}{|\boldsymbol{r} - \boldsymbol{r}'|}.$$

Within an exponential accuracy the quantum decay rate of the metastable state is proportional to

$$\Gamma \propto \exp(-S_E/\hbar)$$
,

where  $S_E$  is the effective Euclidean action calculated at the optimum escape path. This path starts at the entrance point under the potential barrier and ends at the point at which the optimum fluctuation escapes from the barrier [21, 22]. In other words, quantum fluctuation penetrates through the potential barrier along the path of least resistance. Before calculating the quantum rate at which the crystal facet breaks up, we must go over to the effective Euclidean action defined in imaginary time  $t \to it$ . We refer to books [24, 25] for details.

As a result, we should analyze the following functional defined within the time interval  $[-\hbar/2T, \, \hbar/2T]$ 

$$S_E = \int dt \, L_E[\zeta(t, \mathbf{r}), \dot{\zeta}(t, \mathbf{r})],$$

$$L_E = \frac{1}{2} \iint d^2r \, d^2r' \, \frac{\dot{\zeta}(t, \mathbf{r})\dot{\zeta}(t, \mathbf{r}')}{2\pi|\mathbf{r} - \mathbf{r}'|}$$

$$+ \int d^2r \left(\frac{\alpha_1}{\alpha_0}|\nabla\zeta| + \frac{(\nabla\zeta)^2}{2} + \frac{\zeta^2}{2}\right)$$

$$-\frac{1}{2} \iint d^2r \, d^2r' \, \frac{E^2}{(2\pi)^2} \frac{\left(\nabla_{\mathbf{r}}\zeta(t, \mathbf{r})\nabla_{\mathbf{r}'}\zeta(t, \mathbf{r}')\right)}{|\mathbf{r} - \mathbf{r}'|}.$$

Again, the exact determination of extrema for the action  $S_E$  is a rather complicated problem. We here consider only the case of zero temperature when the limits of integration over imaginary time are infinite. As before, it is convenient to take an advantage of the dimensional analysis and variational principle. We will express the surface distortion  $\zeta(t, r)$  in the terms of function f(y, x) with the scaled arguments as

$$\zeta(t, \mathbf{r}) = \zeta f(t/\tau, r/R).$$

Next, we calculate the action  $S_E$  at zero temperature

$$S_{E}(\zeta, \tau, R) = F \frac{\zeta^{2} R^{3}}{2\tau} + \tau \left( \frac{\alpha_{1}}{\alpha_{0}} A |\zeta| R + B \frac{\zeta^{2}}{2} + C \frac{\zeta^{2} R^{2}}{2} - D \frac{E^{2}}{4\pi^{2}} \frac{\zeta^{2} R}{2} \right).$$
(11)

The numerical factors are given by the integrals

$$A = \int_{-\infty}^{\infty} dt \int_{0}^{\infty} |f'(t,r)| 2\pi r \, dr,$$

$$B = \int_{-\infty}^{\infty} dt \int_{0}^{\infty} f'^{2}(t,r) 2\pi r \, dr,$$

$$C = \int_{-\infty}^{\infty} dt \int_{0}^{\infty} f^{2}(t,r) 2\pi r \, dr,$$

$$D = \int_{-\infty}^{\infty} dt \int d^{2}r \, d^{2}r' \frac{\mathbf{r} \cdot \mathbf{r}'}{rr'} \frac{f'(t,r)f'(t,r')}{|\mathbf{r} - \mathbf{r}'|}$$

$$= \int_{-\infty}^{\infty} dt \int_{0}^{\infty} dk \left( \int_{0}^{\infty} dr \, 2\pi r f'(t,r) J_{1}(kr) \right)^{2},$$

$$F = \int_{-\infty}^{\infty} dt \int d^{2}r \, d^{2}r' \frac{\dot{f}(t,r)\dot{f}(t,r')}{2\pi |\mathbf{r} - \mathbf{r}'|}$$

$$= \int_{-\infty}^{\infty} dt \int_{0}^{\infty} \frac{dk}{2\pi} \left( \int_{0}^{\infty} dr \, 2\pi r \dot{f}(t,r) J_{0}(kr) \right)^{2},$$

where  $J_0(x)$  and  $J_1(x)$  are the Bessel function of the first kind.

From the condition of vanishing derivatives in  $\zeta$ , R and  $\tau$  for  $S_E$  we find the following parameters of the quantum critical fluctuation

$$\begin{split} |\zeta_q| &= \frac{A}{\sqrt{BC}} \frac{\alpha_1}{\alpha_0} \left( \sqrt{1 - \frac{7}{16} \frac{E_0^4}{E^4}} + \frac{3}{4} \right) \frac{E^2 E_0^2}{E^4 - E_0^4}, \\ R_q &= \frac{7}{4} \sqrt{\frac{B}{C}} \frac{E_0^2}{E^2} \left( \sqrt{1 - \frac{7}{16} \frac{E_0^4}{E^4}} + 1 \right)^{-1}, \\ \tau_q &= \frac{7}{2\sqrt{2}} \left( \frac{F^2 B}{C^3} \right)^{1/4} \frac{E_0^3}{E\sqrt{E^4 - E_0^4}} \frac{\left( \sqrt{1 - \frac{7}{16} \frac{E_0^4}{E^4}} + \frac{3}{4} \right)^{1/2}}{\sqrt{1 - \frac{7}{16} \frac{E_0^4}{E^4}} + 1}. \end{split}$$

Here  $E_0 = (8\pi^2\sqrt{BC}/D)^{1/2}$  which should coincide for the exact solution with the critical field value, i.e.,  $E_0 = E_c = \sqrt{4\pi}$ . Then we calculate the corresponding value of action  $S_q$  according to

$$S_q = \frac{A}{2} \frac{\alpha_1}{\alpha_0} |\zeta_q| R_q \tau_q$$

at the critical point  $(\zeta_q, R_q, \tau_q)$  representing a saddle point of the functional  $S_E$  (11). Finally, we obtain

$$\begin{split} S_q &= \frac{49A^2}{16\sqrt{2}} \frac{(BF^2)^{1/4}}{C^{7/4}} \frac{\alpha_1^2}{\alpha_0^2} \frac{\left(\sqrt{1 - \frac{7}{16} \frac{E_0^4}{E^4}} + \frac{3}{4}\right)^{3/2}}{\left(\sqrt{1 - \frac{7}{16} \frac{E_0^4}{E^4}} + 1\right)^2} \\ &\times \frac{E_0}{E} \left(\frac{E_0^4}{E^4 - E_0^4}\right)^{3/2}. \end{split}$$

Like the potential barrier height, the action  $S_q$  becomes infinite at the same critical field  $E = E_0$ .

To estimate the numerical coefficients F, A, B, C, and D, we choose a physically expedient trial function  $f(t,r) = \exp[-(t^2 + r^2)]$ . The straightforward calculation gives

$$A = \pi^2$$
,  $B = \frac{\pi^{3/2}}{2^{1/2}}$ ,  $C = \left(\frac{\pi}{2}\right)^{3/2}$ ,  $D = \frac{\pi^3}{2}$ ,  $F = \frac{\pi^2}{4}$ 

with the same ratio  $E_c/E_0$  as in (8).

Let us compare the action  $S_q$  with the Planck constant  $\hbar$ . Introducing a facet capillary length  $\lambda_1 = (\alpha_1/\Delta\rho g)^{1/2}$ , we have in the dimensional units

$$\begin{split} \frac{S_q}{\hbar} &= \frac{49A^2}{16\sqrt{2}} \frac{(BF^2)^{1/4}}{C^{7/4}} \left(\frac{\sqrt{\alpha_1 \alpha_0} \rho_{\text{eff}} \lambda_1^7}{\hbar^2}\right)^{1/2} \\ &\times \frac{\left(\sqrt{1 - \frac{7}{16} \frac{E_c^4}{E^4}} + \frac{3}{4}\right)^{3/2}}{\left(\sqrt{1 - \frac{7}{16} \frac{E_c^4}{E^4}} + 1\right)^2} \frac{E_c}{E} \left(\frac{E_c^4}{E^4 - E_c^4}\right)^{3/2} \\ &\approx 4 \times 10^{21} \frac{\left(\sqrt{1 - \frac{7}{16} \frac{E_c^4}{E^4}} + \frac{3}{4}\right)^{3/2}}{\left(\sqrt{1 - \frac{7}{16} \frac{E_c^4}{E^4}} + 1\right)^2} \frac{E_c}{E} \left(\frac{E_c^4}{E^4 - E_c^4}\right)^{3/2}. \end{split}$$

As is seen, even for the electric fields which are dozens of times larger than the critical one  $E_c$ , the ratio  $S_q/\hbar$  has a giant magnitude so that the crystal surface will remain in the well-defined faceted state for the practically infinite time. At  $E\gg E_c$  we have an estimate

$$S_a/\hbar \approx 2.3 \times 10^{21} (E_c/E)^7$$
.

For strong  $E \gg E_c$  fields, in the dimensional CGSE units the bending deflection  $\zeta_0$  and the typical size of inhomogeneity  $R_0$  are given by

$$|\zeta_q| = 4\pi \frac{7A}{4\sqrt{BC}} \frac{\alpha_1}{E^2} \sim 80 \frac{\alpha_1}{E^2},$$

$$R_q = \frac{7}{8} \sqrt{\frac{B}{C}} \frac{4\pi\alpha_0}{E^2} \sim 15 \frac{\alpha_0}{E^2}.$$

For  $E = 300E_c$ , we find approximately  $R_q \sim 13$  nm and  $|\zeta_q| \sim 5$  nm. The estimate of the tunneling time in the strong  $E \gg E_c$  fields yields

$$\tau_a \approx 10^{-2} (E_c/E)^3$$
 (in seconds).

Again, the small gradient approximation is fulfilled since  $|\nabla \zeta| \sim |\zeta_q|/R_q \sim \alpha_1/\alpha_0 \ll 1$ . Let us compare the plasmon frequency  $\Omega_p$  with the inverse time of tunneling  $\tau_q^{-1}$  in order to justify the metallic approximation. We consider the case of strong fields and take  $k \sim 1/R_q$  as a typical wave vector for the spatial size of the surface distortion. Then, using (9),

$$\Omega_p \tau_q \sim \sqrt{\frac{\rho_{\rm eff}}{\Delta \rho}} \sqrt{\frac{e E_c}{m_e g}} \bigg(\frac{E_c}{E}\bigg)^{3/2} \sim 2 \times 10^7 \bigg(\frac{E_c}{E}\bigg)^{3/2}.$$

Thus, in the fields  $E = 300E_c$  the fulfillment of inequality  $\Omega_p \gg \tau_q^{-1}$  evidences for the favor of the metallic approximation.

## IV. THERMAL-QUANTUM CROSSOVER TEMPERATURE. THE DECAY RATE

Let us turn to the thermal-quantum crossover temperature  $T_q$  which separates the classical thermal activation at  $T > T_q$  from the quantum nucleation mechanism at lower  $T < T_q$  temperatures. Here we estimate the thermal-quantum crossover temperature  $T_q = T_q(E)$  as a ratio of the potential barrier height to the saddle value  $S_q$  of the Euclidean action at zero temperature. In the dimensional units we have then

$$T_q(E) = \frac{\hbar U_0}{S_q} = \hbar \left(\frac{\alpha_0}{\rho_{\rm eff} \lambda_0^3}\right)^{1/2} \frac{8\sqrt{2}}{49} \frac{a^2 C^{7/4}}{c A^2 (BF^2)^{1/4}} \times \frac{\left(\sqrt{1 - \frac{7}{16} \frac{E_c^4}{E^4}} + 1\right)^2}{\left(\sqrt{1 - \frac{7}{16} \frac{E_c^4}{E^4}} + \frac{3}{4}\right)^{3/2}} \frac{E\sqrt{E^4 - E_c^4}}{E_c^3} \approx 1.7 \times 10^{-7} \frac{\left(\sqrt{1 - \frac{7}{16} \frac{E_c^4}{E^4}} + 1\right)^2}{\left(\sqrt{1 - \frac{7}{16} \frac{E_c^4}{E^4}} + \frac{3}{4}\right)^{3/2}} \frac{E\sqrt{E^4 - E_c^4}}{E_c^3} \text{ (in mK)}.$$

Note that the thermal-quantum crossover temperature is independent of the step tension coefficient  $\alpha_1$ . This point is obvious since the barrier height  $U_0$  and action  $S_q$  are both proportional to the same factor  $\alpha_1^2$ .

At the electric fields comparable with the critical one  $E_c$  the thermal-quantum crossover temperature, starting from its zero value at  $E=E_c$ , proves to be extremely small. In the strong  $E\gg E_c$  fields the thermal-quantum crossover temperature grows approximately as a cube of the field

$$T_q(E) \approx 3 \times 10^{-7} (E/E_c)^3$$
 (in mK).

For fields  $E=300E_c$ , we may expect a reasonable magnitude for the thermal-quantum crossover temperature of about 8 mK.

Let us consider a charged crystal facet prepared in the metastable  $E > E_c$  state with adjusting thermodynamic parameters such as temperature T and electric field E. After the lapse of some time  $t_{obs}$ , there will appear a nucleus of the rough state breaking the crystal faceting. Then the nucleation rate  $\Gamma = \Gamma(T, E)$  and the time of observation  $t_{obs}$  are connected by the following relation

$$t_{obs}N_{nuc}\Gamma \simeq 1$$
,

where  $N_{nuc}$  is the total number of independent nucleation sites and  $\Gamma$  is the nucleation rate at a single nucleation site. We estimate  $N_{nuc}$  approximately as the total number of atoms at the crystal surface, assuming that every atom at the surface has an equal possibility to become a nucleation site within the time interval  $t_{obs}$ . For the crystal area of 1 cm<sup>2</sup>, we put

$$N_{nuc} \sim 10^{14}$$
.

The nucleation rate  $\Gamma$  can approximately be estimated as

$$\Gamma \sim \nu \exp(-S)$$

where  $\nu$  is the attempt frequency and exponent S, depending on temperature, is either Arrhenius exponent  $U_0/T$  or Euclidean one  $S_q/\hbar$ . The attempt frequency  $\nu$  is associated with the surface fluctuations resulting in nonzero bending  $\zeta(t, \mathbf{r})$  of the flat crystal facet. In general, the frequency of crystal surface fluctuations depends on the magnitude of surface bending  $\zeta$  and the radius of deformation R as well. This frequency can be estimated by equating the kinetic energy to the potential surface energy in Lagrangian L (10). The order-of-magnitude estimate can be represented as [26]

$$\nu \sim \left(\frac{\alpha_1 R + \alpha_0 |\zeta|}{\rho_{\text{eff}} |\zeta| R^3}\right)^{1/2}.$$

According to [27], there is one optimum path, i.e., escape path which connects the entrance point with the optimum escape point and corresponds to the saddle-point value of the effective Euclidean action. In the quasiclassical approximation the main contribution to the decay rate of the metastable state is determined by such optimum escape path and its nearest vicinity. As is found above, at the optimum escape path a ratio of surface deformation  $\zeta$  to its radius R satisfies approximately  $|\zeta|/R \sim \alpha_1/\alpha_0$ . Then we arrive at

$$\nu \sim \left(\frac{\alpha_1^3}{\alpha_0^2 \rho_{\text{eff}} |\zeta|^3}\right)^{1/2}.$$

Next, we should estimate the equilibrium fluctuations of the surface bending as a function of temperature. At high temperatures one expects the thermal activation mechanism when the average energy fluctuations should be of the order of the temperature, i.e.,  $\alpha_1 R|\zeta| + \alpha_0 \zeta^2 \sim \alpha_0 \zeta^2 \sim T$ . Hence, for T = 1 K, we expect

$$|\zeta| \sim 0.3 \,\mathrm{nm}, \ R \sim 3 \,\mathrm{nm} \ \mathrm{and} \ \nu \sim 5 \times 10^{10} \,\mathrm{Hz}.$$

At zero temperature the attempt frequency can be associated with the zero-point oscillations in the same potential  $U = \alpha_1 R|\zeta| + \alpha_0 \zeta^2 \sim \alpha_0 \zeta^2$ . Using  $U \sim \hbar \nu(U)$  for an estimate of the ground level energy, we find

$$\nu \sim \left(\frac{\alpha_1^6}{\alpha_0 \hbar^3 \rho_{\rm off}^2}\right)^{1/7} \sim 7 \times 10^{10}\,{\rm Hz}, \quad U = \hbar \nu \sim 0.5\,{\rm K}.$$

Note that the magnitude of the surface bending is about of the interatomic spacing and the frequency has numerically the same order of the magnitude as the Debye frequency. These magnitudes seem us reasonable. Thus, we have a relatively large preexponential factor

$$\nu N_{nuc} \sim 10^{24} \,\mathrm{s}^{-1} \sim e^{55} \,\mathrm{s}^{-1}$$

which can readily be compensated by macroscopically large potential barrier for insufficiently high density of charges. Eventually, if we wish to discover the process of the facet destruction for the time of about tens seconds, the exponents  $U_0/T$  or  $S_q/\hbar$  should be kept about 55.

Due to strong exponential dependence of nucleation rate  $\Gamma$  on the thermodynamic parameters T and E the statistical dispersion of nucleation events is not large as compared with the average values of the thermodynamic parameters at which the nucleation is mainly observed. The overwhelming majority of experimental points will concentrate in the narrow region around the average values which correspond to the so-called rapid nucleation line. In essence, from the viewpoint of the time of observation the rapid nucleation line separates the metastable states into two region. One region represents the long-living states looking as stable during the experiment and the other is the short-living states which decay practically instantly.

So, for the rapid nucleation line or the breaking field  $E_b$ , we may expect the following behavior. Under thermal activation mechanism at high temperatures one should observe

$$E_b(T) \propto T^{-1/4}, \quad T > T_q.$$

Below the thermal-quantum crossover temperature this behavior should go over to the practically temperature-independent behavior

$$E_b(T) \approx \text{const}, \quad T < T_q.$$

In the latter connection we would like to mention a possible effect of the energy dissipation processes. As is known from the quantum dynamics of first-order phase transitions [28], the energy dissipation processes increase the effective Euclidean action and thus reduce the quantum decay rate. Accordingly, the behavior of the breaking field  $E_b(T)$  in the quantum regime should grow with the temperature rise and demonstrate a maximum at the thermal-quantum crossover temperature. However, as is mentioned above, the energy dissipation in superfluid <sup>4</sup>He is not large at low temperatures because of negligible density of the normal component. That is why, we expect only a slight manifestation of the energy dissipation effects in the quantum regime.

### V. SUMMARY

To summarize, we have examined a stability of the charged crystal  $^4$ He surface in the atomically smooth and well-faceted state below the roughening transition temperature. Like the charged crystal  $^4$ He surface in the rough state, the charged crystal  $^4$ He facet becomes unstable at the same density of charges or corresponding critical electric field  $E_c$ . However, the dynamics of the transition from the initial homogeneous distribution of charges and flat crystal surface to a spatially unhomogeneous charge distribution and to a warped crystal surface proves to be qualitatively different.

In the rough surface state, as the electric field exceeds the critical value  $E_c$ , the homogeneous surface state becomes absolutely unstable and in this sense the development of the charge-induced instability resembles a

second-order phase transition. In contrast, as the electric field exceeds the same critical value  $E_c$ , the homogeneous state of the atomically smooth and well-faceted crystal surface is converted into the metastable state separated with a potential barrier governed by the electric field or charge density. The barrier height is proportional to the square of the linear facet step energy.

The onset and development of the charge-induced instability at the crystal facet can be compared with the kinetics of first-order phase transitions accompanied by the nucleation and next growth of new stable phase. A nucleus of new phase here can be described as a fluctuation region of the crystal surface in the atomically rough state. The larger the charge density, the smaller the radius of the critical nucleus.

Unlike the charged rough crystal surface, in the electric fields which are tens times larger than the critical value  $E_c$  the potential barrier still remains so high that the

charged crystal facet, though metastable, will not break up for the experimentally obtainable uptime. To realize a breakage of the crystal facet, a few hundreds of critical value  $E_c$  should be achieved. The breaking electric strength  $E_b$  depends on the temperature as well as the nature of the breaking mechanism. At high temperatures the breaking dynamics is associated with the thermal activation mechanism and with the quantum tunneling through a potential barrier at sufficiently low temperatures.

### ACKNOWLEDMENTS

The author is thankful to V. L. Tsymbalenko for stimulating discussion. The work is supported in part by the RFBR Grant No. 10-02-00047a.

- [1] P. Leiderer, J. Low Temp. Phys. 87, 247 (1992).
- [2] L. P. Gorkov and D. M. Chernikova, Pisma ZhETF 18, 119 (1973) [JETP Lett. 18, 68 (1973)].
- [3] K. Mima and H. Ikezi, Phys. Rev. B 17, 3567 (1978).
- [4] P. Leiderer, Phys. Rev. B 20, 4511 (1979); Physica B+C 126, 92 (1984).
- [5] W. Ebner and P. Leiderer, Phys. Lett. A 80, 277 (1980).
- [6] H. Ikezi, Phys. Rev. Lett. 42, 1688 (1979).
- [7] V. B. Shikin and P. Leiderer, Zh. Eks. Teor. Fiz. 81, 184 (1981) [Sov. Phys. JETP 54, 92 (1981)].
- [8] V. I. Melnikov and S. V. Meshkov, Zh. Eksp. Teor. Fiz. 81, 951 (1981) [Sov. Phys. JETP 54, 505 (1981)].
- [9] V. I. Melnikov and S. V. Meshkov, Zh. Eksp. Teor. Fiz. 82, 1910 (1982) [Sov. Phys. JETP 55, 1099 (1982)].
- [10] D. Savignac and P. Leiderer, Phys. Rev. Lett. 49, 1869 (1982).
- [11] M. Uwaha and P. Nozièrez, J. Phys. (Paris) 47, 263 (1986).
- [12] P. Leiderer, Z. Phys. B **98**, 303 (1995).
- [13] J. Bodensohn, K. Nicolai, and P. Leiderer, Z. Phys. 64, 55 (1986).
- [14] A. F. Andreev and A. Ya. Parshin, Zh. Eksp. Teor. Fiz. 75, 1511 (1978) [Sov. Phys. JETP 48, 763 (1978)].
- [15] K. O. Keshishev, A. Ya. Parshin, and A. V. Babkin, Zh. Ekps. Teor. Fiz. 80, 716 (1981) [Sov. Phys. JETP 53, 362 (1981)].
- [16] S. Balibar, H. Alles, and A. Ya. Parshin, Rev. Mod. Phys. 77, 317 (2005).

- [17] L. D. Landau, The Equilibrium Form of Crystals, in Collected Papers (Pergamon, Oxford, 1965).
- [18] P. Nozières, in Solids Far From Equilibrium, edited by C. Godrèche (Cambridge University Press, Cambridge, 1992), p. 38.
- [19] V. L. Tsymbalenko, J. Low Temp. Phys. 88, 55 (1992).
- [20] J. P. Ruutu, P. J. Hakonen, J. S. Penttila, and A. V. babkin, J. P. Saramaki, E. B. Sonin, Phys. Rev. Lett. 77, 2514 (1996).
- [21] S. N. Burmistrov, L. B. Dubovskii, and V.L. Tsymbalenko, J. Low Temp. Phys. 90, 363 (1993).
- [22] E. Tanaka, K. Hatakeyama, S. Noma, S. N. Burmistrov, and T. Satoh, J. Low Temp. Phys. 127, 81 (2002).
- [23] S. Balibar, J. Low Temp. Phys. **129**, 363 (2002).
- [24] Quantum Tunneling in Condensed Media, edited by Yu. Kagan and A. J. Leggett (Elsevier, Amsterdam, 1992)
- [25] U. Weiss, Quantum Dissipative Systems in Series in Modern Condensed Matter Physics (World Scientific, Singapore, 1999), vol. 10.
- [26] S. N. Burmistrov, Fiz. Nizk. Temp. 37, 477 (2011) [Low Temp. Phys. 37, 372 (2011)].
- [27] A. Schmid, Ann. Phys. 170, 333 (1985).
- [28] S. N. Burmistrov and L. B. Dubovskii, Zh. Eksp. Teor.
   Fiz. 93, 733 (1987) [Sov. Phys. JETP 66, 414 (1987)];
   Phys. Lett. A 127, 79 (1988).